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ADDENDUM

Addendum to 'On the singularity structure of the 2D Ising model susceptibility'

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Abstract. A remarkable product formula first derived by Palmer and Tracy (1981 *Adv. Appl. Math.* **2** 329) for the integrand of the two-dimensional Ising model susceptibility expansion coefficients $\chi^{(2n)}$ for temperatures T less than the critical T_c is shown to apply equally for $\chi^{(2n+1)}$ for $T > T_c$ and agrees with formulae derived by Yamada (1984 *Prog. Theor. Phys.* **71** 1416). This new representation simplifies the derivation of the results in the original paper of this title (1999 *J. Phys. A: Math. Gen.* **32** 3889) to the extent that the leading series behaviour and the singularity structure can be deduced almost by inspection. The derivation of series is also simplified and I show, using extended series and knowledge of the singularity structure, that there is now unambiguous evidence for correction to scaling terms in the susceptibility beyond those inferred from a nonlinear scaling field analysis.

1. Introduction

In a recent paper, hereafter referred to as I [1], I reduced the exact formal integral expressions for the dispersion series coefficients $\chi^{(2n+1)}$ of the high-temperature, $T > T_c$, susceptibility of the two-dimensional (2D) square lattice Ising model [2] to the point where one could make definitive statements about both the leading-order series behaviour in the 'temperature' variable $s = \sinh(2K)$ and the singularity structure as a function of complex s. The conclusion, supported by some series analysis, was that the unit circle $|s| = 1$ is very likely to be a natural boundary for the susceptibility χ_+. I subsequently became aware of a truly remarkable simplification by Palmer and Tracy [3] for the susceptibility χ_- in the low-temperature ordered phase $T < T_c$ or equivalently $s > 1$. These authors have shown that a Pfaffian, which is the most difficult factor in the $\chi^{(2n)}$ integrals to evaluate, in fact, reduces to a simple product form. There is no immediately obvious relation to the corresponding factor in $\chi^{(2n+1)}$ in the disordered phase which first, is a sum of Pfaffians and second, has elements that are different from those in the Pfaffians in the low-temperature phase. Surprisingly, however, the high-temperature factor reduces to the *same product form* and the proof of this is given in section 3.

Essentially the same formal expressions, albeit given in terms of Jacobi elliptic functions rather than in a trigonometric/hyperbolic representation, has been reported by Yamada [4] with details of the derivation appearing in a sequence of papers culminating in [5]. The latter reference includes an important appendix that shows how product formulae arise as a general property of certain determinants of elliptic functions. To assist the reader in verifying the equality of the Yamada results with those given here I have collected a short dictionary of mappings, for the most part taken from Yamada [6] and Onsager [7], in appendix C. The fact that the formal results obtained by different methods agree and are consistent with (known) low-order series is strong evidence in favour of their correctness.

Since the susceptibilities, $\chi_\pm$, in the two phases are identical in form, the results reported in I for $T > T_c$ can be directly transcribed to cover $T < T_c$. In particular, there is a corresponding infinity of singularities in χ_-, on the circle $|s| = 1$, which is the analytic continuation from real $T < T_c$, except that these are now branch points of half-integer order rather than logarithmic. A summary of final results including high- and low-temperature comparisons is given in section 2.

Perhaps more important from a practical point of view is the fact that the product representation for the integrands in $\chi^{(N)}$ simplifies the analysis in I dramatically. Series longer by about 30 terms can be obtained with comparable effort and in appendix A I supplement the coefficients given in I to yield complete series to $O(s^{117})$ and $O(s^{-116})$ for the high- and low-temperature cases. Both the long length of the series and knowledge of the singularity structure of $\chi_\pm$ in the complex-s-plane are crucial to enable the simple analysis I present in section 4. I find that corrections to scaling beyond those predicted by the Aharony–Fisher nonlinear scaling field analysis [8] must be present in the ferromagnetic susceptibility. The results are consistent with the leading non-trivial corrections being (amplitude) modifications of existing terms of the form $|t|^{9/4}$ or $t^2 \ln |t|$ where $t = T/T_c - 1$. However, to confidently decide between these possibilities or show that they have not been confused with other nearby power-law terms will, at the very least, require more detailed analysis, possibly similar to that performed by Gartenhaus and McCullough on shorter series [9].

Any future series analysis would be facilitated if one knew in advance what kind of corrections to scaling to expect. Barma and Fisher [10] report numerical evidence, on Ising-like systems with a modified spin distribution, for a correction to a scaling exponent $\theta = \frac{4}{3}$ consistent with a conjecture by Nienhuis [11]. This correction vanishes as the Ising limit is approached; however, Barma and Fisher point out that if the correction couples at third order to a critical operator then corrections $t^4 \ln |t|$ in the scaling field (yielding corrections $|t|^{9/4} \ln |t|$ to χ) are to be expected. Sokal [12] has suggested that an operator associated with the breaking of rotational invariance might, at second order, contribute at the same level. Whether the series derived in this paper are consistent with such a generalized scaling field approach remains an open question. To disentangle $|t|^{9/4} \ln |t|$, $|t|^{9/4}$ and $t^2 \ln |t|$ contributions numerically will be very difficult even with more sophisticated analyses and may in the end require even longer series.

Both the virtue and fault of dealing with the pure Ising model is that there are no parameters coupled to irrelevant variables to vary. A possibly more sensible alternative is to introduce such variables as perturbations; the calculation of χ will then require the evaluation of n-point functions with $n > 2$. Such calculations as series to the necessary high order in s or s^{-1} would not be trivial but if the simplifications described in this paper for χ generalize to these n-point functions then they are not out of the question.

2. Analytical summary

A summary of the results for the susceptibility is as follows. Let $\hat{\chi}^{(N)}$ be reduced expansion coefficients related to $\chi^{(N)}$ from [2] by

$$\beta^{-1}\chi_\pm = \begin{cases} \sum_{n=0}^{\infty} \chi^{(2n+1)} = (1-s^4)^{1/4} s^{-1} \sum_{n=0}^{\infty} \hat{\chi}^{(2n+1)} & s < 1 \\ \sum_{n=1}^{\infty} \chi^{(2n)} = (1-s^{-4})^{1/4} \sum_{n=1}^{\infty} \hat{\chi}^{(2n)} & s > 1. \end{cases} \tag{1}$$

Then (equations in the current text that are the same as, or closely related to, equations in I will also be designated by (I.xx))

$$\hat{\chi}^{(N)} = \frac{1}{N!}\left(\prod_{m=1}^{N-1}\int\frac{\mathrm{d}\phi_m}{2\pi}\right)\left(\prod_{m=1}^{N} y_m\right)(G^{(N)}\{h_{ij}\})^2\left(1+\prod_{m=1}^{N} x_m\right)\bigg/\left(1-\prod_{m=1}^{N} x_m\right) \qquad (2, \mathrm{I}.3)$$

where the constraint $\sum_{m=1}^{N}\phi_m = 0 \bmod 2\pi$ is understood and†

$$\begin{aligned} x_m^{-1} &= s + s^{-1} - \cos\phi_m + ((s+s^{-1}-\cos\phi_m)^2 - 1)^{1/2} = \exp(\gamma_m) \\ y_m^{-1} &= ((s+s^{-1}-\cos\phi_m)^2 - 1)^{1/2} = \sinh(\gamma_m). \end{aligned} \qquad (3, \mathrm{I}.4)$$

The generator $G^{(N)}$ is unity for $N = 1$ and otherwise

$$\begin{aligned} G^{(N)} &= \left(\prod_{m=1}^{N} x_m\right)^{(N-1)/2}\prod_{1\leqslant i<j\leqslant N}(2\sin\tfrac{1}{2}(\phi_i-\phi_j)/(1-x_ix_j)) = \prod_{1\leqslant i<j\leqslant N} h_{ij} \\ h_{ij} &= \sin\tfrac{1}{2}(\phi_i-\phi_j)/\sinh\tfrac{1}{2}(\gamma_i+\gamma_j) = \sinh\tfrac{1}{2}(\gamma_i-\gamma_j)/\sin\tfrac{1}{2}(\phi_i+\phi_j) \end{aligned} \qquad (4)$$

where the equality of the two forms for h_{ij} can be verified using the identity $\cosh\gamma_m = \frac{1}{2}(x_m + x_m^{-1}) = s + s^{-1} - \cos\phi_m$ that follows from the defining equations (3). The product formula (4) for the generator for N even and $s > 1$ was obtained by Palmer and Tracy [3]; Yamada [4] independently derived it for all s. The equivalence of the product formula to the antisymmetric sum,

$$G^{(2n+1)} = \sum_P \delta_P P\left(\prod_{m=1}^{n} f_{2m-1,2m}\right)/(2^n n!) = \sum_{1}^{2n+1}\mathrm{Pf}\{f_{ij}\} \qquad (5, \mathrm{I}.7)$$

with

$$f_{ij} = \tfrac{1}{2}(\sin\phi_i - \sin\phi_j)(1+x_ix_j)/(1-x_ix_j) = \cos\tfrac{1}{2}(\phi_i+\phi_j)\cosh\tfrac{1}{2}(\gamma_i+\gamma_j)h_{ij} \qquad (6, \mathrm{I}.4)$$

for $N = 2n+1$ odd and $s < 1$, given in I will be proved in the following section. The second equality in (5) is a schematic reminder that, as described in I, the permutation sum is over $2n+1$ indices so that $G^{(2n+1)}$ can be expressed as an appropriately signed sum of $2n+1$ Pfaffians $\mathrm{Pf}\{f_{ij}\}$ of order $2n$.

Nappi [13] derived the scaling limit of the antisymmetric sum (5), and the corresponding Pfaffian expression $G^{(2n)} = \mathrm{Pf}\{h_{ij}\}$ for $s > 1$, starting from Wu *et al*'s [2] formulae by a combinatorial route which is essentially that given in I. In fact, since the proof is combinatorial, the reduction first to the scaling limit is not necessary and the results of [13] are general. That (5) might further be reduced to a product form is made plausible by Palmer and Tracy's result ([3], equation (5.20)) that this happens in the scaling limit and indeed Yamada [4] has given such a formula.

Yamada's work [4–6] is a generalization of the spectral approach of Yang [14] and his expressions for the zero-field susceptibility are formally identical to equations (1)–(4). Yamada's formulae (equations (19)–(26) in [4]) are in terms of the Jacobi elliptic functions which are useful for simplifying the integral equation that must be solved in this method, but their transcription to trigonometric/hyperbolic form is easily obtained using the list of elliptic function identities given in appendix C. For numerical work the elliptic representation is probably not useful; as an example, the lattice sum over phases that leads to the trivial constraint $\sum_{m=1}^{N}\phi_m = 0 \bmod 2\pi$ in the trigonometric representation results instead in a very complicated implicit function constraint on the elliptic variables.

† The definition of y_m here differs from that in I by a factor of s to enable a parallel treatment of high and low temperatures.

Abraham (cf [15] and references therein) has also tackled the problem of Ising model correlations by spectral analysis but via a generalization of the fermionic approach of Schultz *et al* [16]. The Pfaffian-like structure of the generators $G^{(N)}$ arises naturally and the method appears to have an advantage in that it eliminates the combinatorial complexity of Wu *et al*'s [2] approach. However, no method is algebraically trivial and the reader is advised to treat Abraham's formulae with caution. In the case of pair correlations at high temperature I find empirically that $u(\boldsymbol{r})$ (equations (18)–(27) in [15]) must be corrected by division by $\sinh^2(K)$. A comparison of the correct $G^{(N)}$ given here with corresponding formulae derived by Abraham is given in appendix B.

For practical computational purposes the most remarkable and useful of the results reported in the literature are the product formulae. An example is Palmer and Tracy's [3] equation (5.8) which, as the second equality in

$$G^{(2n)} = \mathrm{Pf}\{h_{ij}\} = \prod_{1\leqslant i<j\leqslant 2n} h_{ij} \tag{7}$$

expresses the Pfaffian as a product of its elements. With h_{ij} written in the trigonometric/hyperbolic form in equation (4), the result (7) is quite mysterious and indeed the proof [3] required showing that the Pfaffian and product are both elliptic functions and have the same periodicity and singularity structure. Yamada's proof [6] of a product representation for $G^{(2n+1)}$ also relies in an essential way on properties of the Jacobi elliptic functions. Such knowledge is not required for the proof given in section 3 where I start from Palmer and Tracy's result (7) and proceed entirely by algebraic manipulation. The convergence of different methods to the same final result (1)–(4) is important for confirming its validity.

I conclude this section with a short digression on the implications of the product representation of $G^{(N)}$. First, the product formula (4) makes the leading-series behaviour of $\hat{\chi}^{(N)}$ immediately obvious. Since $y_m \simeq s$ (or s^{-1}) and $x_m \simeq y_m/2$ for small s (or s^{-1}), depending on whether the temperature is above (or below) T_c, the leading term in $\hat{\chi}^{(N)}$ is

$$\hat{\chi}^{(N)} \simeq (s\ (\text{or } s^{-1}))^{N^2}/2^{N(N-1)}A_N \tag{8, I.8}$$

by inspection of equations (2)–(4) with A_N the integral

$$A_N = \frac{1}{N!}\left(\prod_{m=1}^{N-1}\int \frac{\mathrm{d}\phi_m}{2\pi}\right)\left|\prod_{1\leqslant i<j\leqslant N} 2\sin\tfrac{1}{2}(\phi_i-\phi_j)\right|^2 = 1. \tag{9}$$

That $A_N = 1$ can be seen by noting that the product over sine functions is, except for an overall phase, a product over the differences $\exp(\mathrm{i}\phi_i) - \exp(\mathrm{i}\phi_j)$ and thus a Vandermonde determinant in the variables $\exp(\mathrm{i}\phi_i)$. This determinant in turn is the sum of $N!$ terms of the form $\pm\exp(\mathrm{i}\sum_{m=1}^{N-1}\phi_m(n_m - n_N))$ with n_i an integer. All cross terms in the product of the determinant with its complex conjugate will vanish when integrated because the n_i do not match; only the $N!$ diagonal terms will survive, each with an integral value of unity.

Secondly, the determination of singularity amplitudes for $T > T_c$ (or $T < T_c$) at

$$\begin{aligned} &s_{kl}\ (\text{or } s_{kl}^{-1}) = \exp(\mathrm{i}\theta_{kl}) \qquad && 2\cos(\theta_{kl}) = \cos(\phi^{(k)}) + \cos(\phi^{(l)}) \\ &\phi^{(k)} = 2\pi k/N && \phi^{(l)} = 2\pi l/N \end{aligned} \tag{10, I.12}$$

proceeds formally exactly as in I except that the hard part of determining the constant $B_{kl}^{(N)}N!$ to be $1/(2\sin(\phi^{(l)}))^{N(N-1)}$ from its defining equation

$$(G^{(N)}\{h_{ij}\})^2/N! \simeq \mathrm{i}^{N(N-1)}B_{kl}^{(N)}\prod_{1\leqslant i<j\leqslant N}(\delta_i-\delta_j)^2 \tag{11, I.39}$$

now follows trivially by inspection of equation (4) with $\gamma_m \simeq -\mathrm{i}\phi^{(l)}$ and the deviation $\delta_m = \phi_m - \phi^{(k)}$. There is a technical distinction between high and low temperatures that must be observed in the actual evaluation of the integrals; power counting shows that for $T < T_\mathrm{c}$ the $|s| = 1$ singularities are branch points of half-integer order. Specifically, let the deviation ϵ for $T < T_\mathrm{c}$ be defined by $s^{-1} = s_{kl}^{-1}(1-\epsilon)$. Then the singular part of $\hat{\chi}^{(N)}$ is†

$$\hat{\chi}_{kl}^{(N)} \simeq (\mathrm{i}\epsilon N \sin(\theta_{kl}))^{(N^2-3)/2}\left(\prod_{m=1}^{N-1}(m!/2^m)\right)\Big/(\pi^{(N-3)/2}\Gamma(\tfrac{1}{2}(N^2-1))\sqrt{N})$$
$$\times(\sin^2(\phi^{(l)})\cos(\phi^{(k)}) + \sin^2(\phi^{(k)})\cos(\phi^{(l)}))^{-(N^2-1)/2} \qquad N \text{ even.} \qquad (12)$$

The phases in equation (12) are given for $0 < \theta_{kl} < \pi/2$; elsewhere they can be inferred by invoking reality and invariance under $s^{-1} \to -s^{-1}$.

3. Proof of the product representation

I now outline the demonstration of the equivalence of the antisymmetric sum equation (5) with the product equation (4). The relevance of Palmer and Tracy's result (7) to the high-temperature regime is that $G^{(2n+1)}$ in equation (5) is a sum of $2n+1$ Pfaffians $\mathrm{Pf}\{f_{ij}\}$ of order $2n$ and, with f_{ij} related to h_{ij} by equation (6),

$$\mathrm{Pf}\{f_{ij}\} = \cos\left(\frac{1}{2}\sum_{m=1}^{2n}\phi_m\right)\cosh\left(\frac{1}{2}\sum_{m=1}^{2n}\gamma_m\right)\mathrm{Pf}\{h_{ij}\}. \qquad (13)$$

To verify equation (13) note that a Pfaffian is a sum of products and any particular product term $\prod f_{ij}$ can be rewritten as

$$\prod\cos\tfrac{1}{2}(\phi_i+\phi_j)\prod\cosh\tfrac{1}{2}(\gamma_i+\gamma_j)\prod h_{ij} = \left(\cos\left(\frac{1}{2}\sum_{m=1}^{2n}\phi_m\right) - \sum\prod \mathrm{trg}\,\tfrac{1}{2}(\phi_i+\phi_j)\right)$$
$$\times\left(\cosh\left(\frac{1}{2}\sum_{m=1}^{2n}\gamma_m\right) - \sum\prod\mathrm{trgh}\,\tfrac{1}{2}(\gamma_i+\gamma_j)\right)\prod h_{ij} \qquad (14)$$

where trg denotes either sine or cosine and trgh the corresponding hyperbolic functions. The precise form of the sum of products $\sum\prod$ trg is unimportant except that each term contains at least two sine factors; similarly for the hyperbolic term. The leading term in equation (14) gives the equation (13) result we wish to prove; all remaining terms containing $\sum\prod$ trg and/or $\sum\prod$ trgh vanish when $\prod f_{ij}$ is summed to generate the Pfaffian. The sine and/or sinh factors play a crucial role in this. The essential point is that these factors, as multipliers of the corresponding h_{ij}, eliminate the denominators in h_{ij}. That is, $\sinh\frac{1}{2}(\gamma_i+\gamma_j)h_{ij} = \sin\frac{1}{2}(\phi_i-\phi_j)$ and $\sin\frac{1}{2}(\phi_i+\phi_j)h_{ij} = \sinh\frac{1}{2}(\gamma_i-\gamma_j)$ which are just the definitions (4) rewritten. Since there is at least one pair of these denominator-free factors in every product one can rearrange the Pfaffian sum of correction terms from equation (14) so as to contain only terms of the form

$$\delta\mathrm{Pf}\{f_{ij}\} = \{\hat{h}_{ij}\hat{h}_{kl} - \hat{h}_{ik}\hat{h}_{jl} + \hat{h}_{il}\hat{h}_{jk}\}S_{ijkl} \qquad (15)$$

† For N odd, equation (I.14) is recovered by multiplying $\hat{\chi}_{kl}^{(N)}$ in equation (12) by $-\ln\epsilon/\pi$. A formula applicable for both odd and even N, namely $\Delta\hat{\chi}_{kl}^{(N)}(\epsilon) = 2\mathrm{i}\hat{\chi}_{kl}^{(N)}(-\epsilon)$ with $\hat{\chi}_{kl}^{(N)}$ given by equation (12), describes the discontinuity across the cut which is chosen as real, negative ϵ.

where S_{ijkl} is a function that is symmetric in i, j, k, l and $\hat{h}_{ij}$ is one of $\sin\frac{1}{2}(\phi_i - \phi_j)$, $\sinh\frac{1}{2}(\gamma_i - \gamma_j)$, $2\cos\frac{1}{2}(\phi_i + \phi_j)\sin\frac{1}{2}(\phi_i - \phi_j) = \sin(\phi_i) - \sin(\phi_j)$ or $\cos(\phi_i) - \cos(\phi_j)$. In all cases the sum multiplying S_{ijkl} in equation (15) vanishes and equation (13) is proved.

To complete the proof of equation (4) in the high-temperature phase, I now replace $\mathrm{Pf}\{h_{ij}\}$ in equation (13) by its product form and utilize the connection between the sine product $\prod_{i<j}\sin\frac{1}{2}(\phi_i - \phi_j)$ and a Vandermonde determinant with elements $\exp(\mathrm{i}\phi_m)$ to obtain the alternative expression

$$\mathrm{Pf}\{f_{ij}\} = \tfrac{1}{2}\left(2\prod_{m=1}^{2n} x_m\right)^{n-1}\left(1 + \prod_{m=1}^{2n} x_m\right)\mathrm{Det}_{2n}(v)\Big/\prod_{1\leqslant i<j\leqslant 2n}(1 - x_i x_j). \tag{16}$$

For the purposes of the subsequent development, the Vandermondian v in equation (16) has been rearranged into real elements $v_{i,j} = \sin(n+1-i)\phi_j$ for $1 \leqslant i \leqslant n$ and $v_{i,j} = \cos(i-n-1)\phi_j$ for $n < i \leqslant 2n$, $1 \leqslant j \leqslant 2n$. Note that v contains a row of elements $\sin(n\phi_j)$ but no corresponding row $\cos(n\phi_j)$. The essence of the remaining argument is to show that $G^{(2n+1)}$ is related to a larger v that contains this $\cos(n\phi_j)$ row.

In detail, let the definitions of v above be extended to include $j = 2n+1$ but for now leave $v_{2n+1,j}$ unspecified. The determinant $\mathrm{Det}_{2n}(v)$ in (16) can be viewed as the cofactor $V_{2n+1,2n+1}$ of this larger matrix and more generally the $2n+1$ sum of Pfaffians which defines $G^{(2n+1)}$ in equation (5) is the column expansion of the determinant $\mathrm{Det}_{2n+1}(v)$. Specifically,

$$\begin{aligned} G^{(2n+1)} &= \left(\prod_{m=1}^{2n+1} x_m\right)^{n}\left(\sum_{m=1}^{2n+1} v_{2n+1,m}V_{2n+1,m}\right)\Big/\prod_{1\leqslant i<j\leqslant 2n+1}(1 - x_i x_j) \\ v_{2n+1,m} &= \tfrac{1}{4}(2/x_m)^n\left(1 + x_m\Big/\prod_{i=1}^{2n+1} x_i\right)\prod_{i\neq m}(1 - x_i x_m) \end{aligned} \tag{17}$$

and of course this definition of $v_{2n+1,m}$ in equation (17) can be modified by the addition of terms proportional to $v_{i,m}$, $1 \leqslant i \leqslant 2n$, with the coefficient of proportionality being any symmetric function of all $2n+1$ variables. In view of this one finds, by explicit expansion, the equivalent expressions

$$\begin{aligned} v_{2n+1,m} &\equiv 2^{n-1}(x_m^n + x_m^{-n}) \equiv 2^{n-1}(x_m + x_m^{-1})^n \equiv 2^{n-1}(-2\cos(\phi_m))^n \\ &\equiv (-2)^n\cos(n\phi_m) \end{aligned} \tag{18}$$

obtained by dropping any terms that might give rise to $\cos(n'\phi_m)$ with $n' < n$. The last equivalence in (18) gives

$$\mathrm{Det}_{2n+1}(v) = \sum_{m=1}^{2n+1} v_{2n+1,m}V_{2n+1,m} = \prod_{1\leqslant i<j\leqslant 2n+1}\left(2\sin\tfrac{1}{2}(\phi_i - \phi_j)\right) \tag{19}$$

if one again invokes the connection between the sine product and the Vandermondian. With the result (19), the expression for $G^{(2n+1)}$ in (17) becomes the equation (4) we wished to verify.

In summary, equations (1)–(4) provide a very simple expression for the 2D Ising model susceptibility. One can speculate that similar simplifications will be found for n-point functions with $n > 2$.

4. Corrections to scaling

Aharony and Fisher [8] have shown that in general, independent of the presence or absence of irrelevant scaling fields, there is a class of corrections to scaling terms that can be eliminated by a simple analytic transformation of the conventional thermal and ordering fields. In the case of the square lattice Ising model I will choose as the thermal and ordering fields

$$\tau = \tfrac{1}{2}(s^{-1} - s) \qquad h = \beta H \tag{20}$$

where the magnetic field H has been normalized such that the magnetization $M = -\partial F/\partial H$ is just the mean spin with its maximum absolute value chosen as unity. The use of τ in equation (20) rather than $t = T/T_c - 1$ simplifies the subsequent formulae but is otherwise of no significance. It is worth noting that at linear order, $\tau = 2K_c\sqrt{2}t$, $2K_c = \ln(1+\sqrt{2})$. The nonlinear scaling fields are

$$g_\tau = \tau g_\tau^{(0)} + \pi E_0/(4K_c\sqrt{2})h^2 g_\tau^{(2)} + \mathrm{O}(h^4) \qquad g_h = h g_h^{(1)} + \mathrm{O}(h^3) \tag{21}$$

where the $g_\tau^{(n)}$ and $g_h^{(n)}$ are functions of τ normalized to unity at $\tau = 0$. On the assumption that irrelevant scaling fields are not present in the square lattice Ising model, the 'optimal' $g_\tau^{(0)}$ is determined by the condition that the singular part of the free energy at zero field scales exactly as $(g_\tau)^2 \ln|g_\tau|$. A short calculation, given the Onsager solution, then yields

$$g_\tau^{(0)} = \left[\int_0^1 \mathrm{d}x\, F(\tfrac{1}{2}, \tfrac{1}{2}; 1; -x\tau^2)/(1+x\tau^2)^{1/2}\right]^{1/2} = 1 - 3\tau^2/16 + 137\tau^4/1536 - \cdots \tag{22}$$

where F is the hypergeometric function. Similarly, the defining equation for the 'optimal' $g_h^{(1)}$ is the scaling of the magnetization in zero field, namely $M_0 = (1-s^{-4})^{1/8} = g_h^{(1)}(-4\tau g_\tau^{(0)})^{1/8}$, from which

$$\begin{aligned} g_h^{(1)} &= [(1+\tau^2)^{1/2}[(1+\tau^2)^{1/2}+\tau]^2/g_\tau^{(0)}]^{1/8} \\ &= 1 + \tau/4 + 15\tau^2/128 - 9\tau^3/512 - 4333\tau^4/98\,304 + \cdots \end{aligned} \tag{23}$$

follows. The singular part of the zero-field susceptibility is

$$\beta^{-1}\chi_\pm = C_{0\pm}(2K_c\sqrt{2})^{7/4}|\tau|^{-7/4}(g_h^{(1)})^2/(g_\tau^{(0)})^{7/4} + E_0/(2K_c\sqrt{2})\tau\ln|\tau| g_\tau^{(0)} g_\tau^{(2)} \tag{24}$$

where $E_0 \approx 0.040\,325\,5003$ [17] and 40-digit accurate values for $C_{0\pm}$ can be found in I. The presence of the second term in equation (24) was a significant prediction of the nonlinear scaling field analysis by Aharony and Fisher [8], but one should note that while $g_\tau^{(0)}$ and $g_h^{(1)}$ as series in τ have finite radii of convergence, $g_\tau^{(2)}$ is at best asymptotic because $|s| = 1$ is a natural boundary for $\chi_\pm$. The formula (24) has been verified numerically through order $\tau^{5/4}$ by Gartenhaus and McCullough [9], but nothing beyond $g_\tau^{(2)} = 1$ could be said because of the limited length of the series available to them.

While equation (24) is a definition of the function $g_\tau^{(2)}$, the fact that it must apply both above and below the critical point allows a check of the Aharony–Fisher analysis. More accurately, a failure of equation (24) indicates the presence of irrelevant scaling fields.

For temperatures $T < T_c$ I test equation (24) by first rewriting each term as a series in $\omega = 1 - s^{-2}$, a definition which combined with equation (20) gives the transformations

$$\tau = -\tfrac{1}{2}\omega/(1-\omega)^{1/2} \qquad \omega = -2\tau[(1+\tau^2)^{1/2} + \tau]. \tag{25}$$

The series expansion of equation (24) in ω is now truncated at some moderate order, re-expanded in series $\sum f_{2m}s^{-2m}$, and finally used to form the difference series $\sum g_{2m}s^{-2m} = \sum(K_{2m} - f_{2m})s^{-2m}$ where the K_{2m} are the known coefficients of $\beta^{-1}\chi_-$ from appendix A. However, before the difference coefficients can be sensibly interpreted one must reduce the effect of the unphysical singularities on the circle $|s^{-2}| = 1$.

The most important singularity is at $s^{-2} = -1$. In the vicinity of this point the dominant contribution to $\beta^{-1}\chi_-$ from $\chi^{(2)}$ is $2^{1/4}/(6\pi)/(1+s^{-2})^{3/4}$ and from $\chi^{(4)}$ is

$-2^{1/4}(\ln(1+s^{-2})+3.067\,58\underline{4})(2G-1)/(16\pi^3)/(1+s^{-2})^{3/4}$ where $G = 0.915\ldots$ is Catalan's constant and the constant additive to the logarithm is a numerical estimate. These two terms by themselves would imply an asymptotic contribution to the coefficient K_{2m} of s^{-2m} in the series for $\beta^{-1}\chi_-$ of magnitude

$$K_{2m} = (-1)^m m^{-1/4} 2^{1/4}/\Gamma(\tfrac{3}{4})(1/(6\pi) + (\ln(m) - \psi(\tfrac{3}{4}) - 3.067\,58\underline{4})(2G-1)/(16\pi^3))$$
$$\approx (-1)^m m^{-1/4}(0.048\,259\,00\underline{0} + 0.001\,627\,3895\ln(m)). \tag{26}$$

The complete contribution to $\beta^{-1}\chi_-$ from all $\chi^{(N)}$ is close to this value. I find almost complete elimination of the $s^{-2} = -1$ singularity effects is possible by a combination of the subtraction

$$K_{2m} \to K_{2m} - (-1)^m m^{-1/4}(0.048\,181\,\underline{5010} + 0.001\,641\,5\underline{55\,38}\ln(m)) \tag{27}$$

and the reduction in amplitude of higher-order terms by repeated use of a smoothing operation D_a as described in I, namely the averaging $D_a g_n = (g_{n-1} + g_{n+1})/2$. For example, $n^{15/4}D_a^3 n^3 D_a^3 n^{13/4} g_n$ will eliminate terms g_{2m} proportional to $(-1)^m m^{-p}\ln(c\cdot m)$, $p = \frac{5}{4}, \frac{9}{4}$ or $\frac{13}{4}$ and c any constant, and convert $(-1)^m m^{-17/4}\ln(c\cdot m)$ to $\mathrm{O}(m^{-1/4}\ln(c\cdot m))$, while at the same time enhancing any non-oscillatory terms associated with the $s^{-1} = 1$ singularity by a factor of n^{10}. It is worth noting that because of the clear numerical evidence of a confluent logarithmic term in equation (27), the naive scaling at the point $s^{-2} = -1$ reported previously [18, 19] is incorrect.

The only complex singularities of any consequence are four symmetry-related points $s^{-1} = \pm\exp(\pm\mathrm{i}\pi/3)$ arising from $\hat{\chi}^{(4)}$. The integrand in equation (2) is sufficiently simple that with algebraic packages such as *Maple*† one can go beyond the leading contribution given in equation (12). The first few singular terms near $s^{-1} = \exp(\mathrm{i}\pi/3)$ written in terms of $\epsilon = 1 - s^{-1}\exp(-\mathrm{i}\pi/3)$ are

$$\hat{\chi}^{(4)} = -3^{1/4}1536/(5005\pi)(1+\mathrm{i})\epsilon^{13/2}\left[1 + \left(\frac{13}{4} + \frac{5}{2\sqrt{3}}\mathrm{i}\right)\epsilon + \left(\frac{1265}{1632} + \frac{25\sqrt{3}}{8}\mathrm{i}\right)\epsilon^2 - \left(\frac{5365}{384} - \frac{53\,321}{20\,672\sqrt{3}}\mathrm{i}\right)\epsilon^3 + \cdots\right]. \tag{28}$$

The dominant effect of the terms in equation (28), combined with their symmetry-related equivalents, can be eliminated by the subtraction

$$K_{2m} \to K_{2m} - 3^{3/8}81/(8\pi^{3/2})m^{-15/2} \times\left[\left(1 + \frac{15}{8}m^{-1} - \frac{9575}{128}m^{-2} - \frac{185\,155}{1024}m^{-3} + \cdots\right)\sin\left(\tfrac{2}{3}m\pi + \tfrac{5}{24}\pi\right) + \frac{5\sqrt{3}}{2}\left(m^{-1} + \frac{17}{8}m^{-2} - \frac{17\,571}{128}m^{-3} + \cdots\right)\cos\left(\tfrac{2}{3}m\pi + \tfrac{5}{24}\pi\right)\right] \tag{29}$$

while higher-order terms can be reduced in magnitude by a smoothing $D_b g_n = (g_{n-2} + g_n + g_{n+2})/3$.

I obtain estimates for the unknown coefficients in $\tau g_\tau^{(0)} g_\tau^{(2)}$ in equation (24) by a least-squares procedure that minimizes the residual

$$R_n = n^{-5/2} D_b^2 n^{25/4} D_a^3 n^3 D_a^3 n^{13/4} g_n. \tag{30}$$

† Maple V software available from Waterloo Maple Software, 160 Columbia Street West, Waterloo, Ontario, Canada N2L 3L3.

A particular fit, following the subtractions given in equations (27) and (29) and in which the first (known) term in equation (24) has been truncated at $\omega^{53/4}$ and the second (unknown) term at $\omega^{10}\ln|\omega|$, is

$$\tau\ln|\tau|g_\tau^{(0)}g_\tau^{(2)} \approx -\tfrac{1}{2}\omega\ln|\omega|(1 + 0.250\,06\underline{0\,119\,086\,839}\omega + 0.107\,\underline{401\,099\,2494}\omega^2 + 0.13\underline{7\,795\,668\,44}\omega^3 + 0.\underline{503\,783\,9523}\omega^4 + \underline{1.565\,146\,753}\omega^5 + \underline{3.295\,256\,38}\omega^6 + \underline{4.272\,8663}\omega^7 + \underline{3.007\,152}\omega^8 + \underline{0.863\,27}\omega^9). \tag{31}$$

As can be explicitly verified, this fit leaves the residuals defined in equation (30) satisfying $|R_n| < 0.1$ on the fitting interval $74 \leqslant n \leqslant 104$. By comparing different truncations and slightly different procedures, I conclude that the underlined digits in equations (27) and (31) are uncertain; however, these digits are highly correlated and must be kept when calculating the residual†. Because of this correlation it is also necessary to keep high-order terms in any fit such as those in equation (31) which apparently are of no significance, yet if forced to vanish would contaminate the low-order terms of interest. In principle, another reason for the lack of significance of the higher-order terms in equation (31) is the limited accuracy with which the amplitude E_0 in equation (24) is known. This, however, is less important than the uncertainties in the current fitting procedure and does not affect any of the conclusions in this paper.

The contribution $g_\tau^{(2)}$ to the nonlinear thermal scaling field (21) deduced from equation (31) is

$$g_\tau^{(2)} \approx 1 + 0.499\,88\underline{0}\tau + 0.116\underline{9}\tau^2 - 0.7\underline{2}\tau^3 + \underline{5}\tau^4 - \cdots \qquad (T < T_c) \tag{32}$$

and this is to be compared to the estimate that is obtained from the series for $T > T_c$ as discussed below.

For the $T > T_c$ analysis I use the ferromagnetic variable $\omega_f = 1 - s$ which is related to the thermal field τ defined in equation (20) by

$$\tau = \omega_f(1-\omega_f/2)/(1-\omega_f) \qquad \omega_f = 2\tau/(1+\tau+(1+\tau^2)^{1/2}). \tag{33}$$

Otherwise the analysis parallels that for $T < T_c$. In particular, the series expansion of equation (24), now in terms of ω_f, is truncated at some moderate order, re-expanded in series $\sum f_n s^n$, and used to form the difference series $\sum g_n s^n = \sum (K_n - f_n)s^n$ with K_n the coefficients of $\beta^{-1}\chi_+$. Contributions to the series in s coming from unphysical singularities on the circle $|s| = 1$ are reduced as described for $T < T_c$. One new situation arises because of the antiferromagnetic singularity at $s = -1$.

This singular point has been treated in detail by Burnett and Gartenhaus [20], who show that in the absence of irrelevant variables, the singular part of the susceptibility can be written as

$$\beta^{-1}\chi_a = F_0/(2K_c\sqrt{2})\omega_a\ln|\omega_a|\phi_a(\omega_a) \qquad \omega_a = 1 + s \tag{34}$$

where $F_0 = -0.193\,595\,186\,268\,2647$ [21] and $\phi_a(\omega_a)$ is an analytic function. They also provide the numerical estimate

$$\phi_a \approx 1 + 0.999\,81(6)\omega_a + 0.903(3)\omega_a^2 \tag{35}$$

and it is easy to confirm their result by generalizing the least-squares fitting procedure to include both ferromagnetic and antiferromagnetic scaling polynomials. A particular fit is

$$\phi_a \approx 1 + 0.999\,\underline{987\,719\,9992}\omega_a + 0.91\underline{4\,979\,169\,25}\omega_a^2 + 1.\underline{359\,885\,44}\omega_a^3 + \underline{4.419\,735}\omega_a^4 + \underline{15.9656}\omega_a^5 + \underline{35.632}\omega_a^6 + \underline{31.31}\omega_a^7 \tag{36}$$

† Because of the overall factor of n^{10} in the definition of R_n in equation (30), small terms in g_n are greatly magnified. For example, a fractional error of as little as 10^{-20} in the series coefficient K_{100} is significant.

where underlined digits are uncertain, as deduced from a comparison of different fits, but have been kept because of correlations between the terms and also with terms in the ferromagnetic polynomial in equation (43) below. The coefficient of ω_a can plausibly be assumed to be unity; if this is imposed then a typical fit becomes

$$\phi_a \approx 1 + \omega_a + 0.917\,\underline{058\,101\,37}\omega_a^2 + 1.4\underline{68\,081\,710}\omega_a^3 + \underline{6.782\,3520}\omega_a^4 + \underline{39.737\,890}\omega_a^5 + \underline{143.076\,89}\omega_a^6 + \underline{205.2129}\omega_a^7. \qquad (37)$$

The rapid growth of the coefficients of ω_a^n with order in equations (36) and (37) is somewhat of a surprise since the nearest known unphysical singularities (of substantial amplitude) to $s = -1$ are at a distance $d = |\exp(\pm 2\pi\mathrm{i}/3) + 1| = 1$. On the other hand, this behaviour and the fact that the new estimates of ϕ_a have drifted beyond the error estimates of equation (35) are what would be expected if the susceptibility is not of the assumed form of equation (34). Thus the current results suggest the presence of irrelevant variables at the antiferromagnetic point but I have not pursued this further. The coefficients in equations (36) and (37) should simply be viewed as 'effective' constants that serve the purpose of reducing the odd–even oscillations in the susceptibility series to the point where a reasonable analysis of the ferromagnetic singularity is possible.

Some useful properties of the important unphysical singularities confounding the ferromagnetic point analysis are as follows. In the vicinity of $s = \pm\mathrm{i}$, the dominant singular contribution to $\beta^{-1}\chi_+$ from $\chi^{(1)}$ is $2^{-3/4}s(1+s^2)^{1/4}$ and from $\chi^{(3)}$ is $2^{-7/4}((2s-\pi)\ln(1+s^2) + 5.636\,2\underline{4}s - 7.692\,0\underline{1})/\pi^2(1+s^2)^{1/4}$ with the constants additive to the logarithm again a numerical estimate. These two terms give, as a leading asymptotic contribution to the coefficient of s^n in $\beta^{-1}\chi_+$,

$$K_n = -n^{-5/4}2^{-3/2}/\Gamma(\tfrac{3}{4})[(1 - (\ln(n/2) - \psi(\tfrac{3}{4}) - 4 - 5.636\,2\underline{4}/2)/\pi^2)\sin(\pi n/2) + ((\ln(n/2) - \psi(\tfrac{3}{4}) - 4 - 7.692\,0\underline{1}/\pi)/(2\pi))\cos(\pi n/2)]. \qquad (38)$$

If the leading terms from $\chi^{(2n+1)}$, $n > 1$, have the same structure then the smoothing operation $D_a^3 n^3 D_a^3 n^{13/4} g_n$, $D_a g_n = (g_{n-1} + g_{n+1})/2$, used in the $T < T_\mathrm{c}$ analysis is appropriate here also.

The remaining important complex singularities are those from $\hat{\chi}^{(3)}$. Near $s = \exp(2\mathrm{i}\pi/3)$, with $\epsilon = 1 - s\exp(-2\mathrm{i}\pi/3)$,

$$\hat{\chi}^{(3)} = 8/(3\pi)\mathrm{i}\epsilon^3\ln(\epsilon)\left[1 + \left(\frac{3}{2} - \frac{5}{2\sqrt{3}}\mathrm{i}\right)\epsilon - \left(\frac{5}{\sqrt{3}}\mathrm{i}\right)\epsilon^2 - \left(\frac{5}{2} + \frac{41}{6\sqrt{3}}\mathrm{i}\right)\epsilon^3 - \left(\frac{55}{9} + \frac{8}{\sqrt{3}}\mathrm{i}\right)\epsilon^4 - \left(\frac{98}{9} + \frac{44}{3\sqrt{3}}\mathrm{i}\right)\epsilon^5 + \cdots\right] \qquad (39)$$

and near $s = (1 + \mathrm{i}\sqrt{15})/4$, with $\epsilon = 1 - s(1 - \mathrm{i}\sqrt{15})/4$,

$$\hat{\chi}^{(3)} = 5\sqrt{5}/(3\pi)\mathrm{i}\epsilon^3\ln(\epsilon)\left[1 + \left(\frac{3}{2} + \frac{47}{8\sqrt{15}}\mathrm{i}\right)\epsilon - \left(\frac{227}{40} - \frac{47}{4\sqrt{15}}\mathrm{i}\right)\epsilon^2 - \left(\frac{267}{16} + \frac{133\,471}{1920\sqrt{15}}\mathrm{i}\right)\epsilon^3 + \left(\frac{949\,957}{18\,432} - \frac{50\,757\sqrt{3}}{640\sqrt{5}}\mathrm{i}\right)\epsilon^4 + \left(\frac{8867\,299}{36\,864} + \frac{75\,566\,527}{122\,880\sqrt{15}}\mathrm{i}\right)\epsilon^5 + \cdots\right]. \qquad (40)$$

The useful series subtractions generated by these singularities and their corresponding complex conjugate points are

$$
\begin{aligned}
K_n \to K_n &- 3^{1/8}\frac{32}{\pi}n^{-4}\Bigg[\left(1-2n^{-1}-\frac{35}{3}n^{-2}+40n^{-3}-\frac{7532}{3}n^{-4}+\frac{29\,554}{3}n^{-5}+\cdots\right)\\
&\times\cos\left(\tfrac{2}{3}n\pi+\tfrac{5}{24}\pi\right)+\frac{4}{\sqrt{3}}\left(2n^{-1}-5n^{-2}+90n^{-3}\right.\\
&\left.-\frac{595}{2}n^{-4}+\frac{15\,176}{3}n^{-5}+\cdots\right)\sin\left(\tfrac{2}{3}n\pi+\tfrac{5}{24}\pi\right)\Bigg]\\
&-15^{1/8}\frac{10\sqrt{10}}{\pi}n^{-4}\Bigg[\left(1-2n^{-1}-\frac{1009}{12}n^{-2}+\frac{1029}{4}n^{-3}+\frac{35\,211\,729}{1280}n^{-4}\right.\\
&\left.-\frac{106\,792\,147}{960}n^{-5}+\cdots\right)\cos\left(\left(n+\tfrac{1}{2}\right)\cos^{-1}\left(\tfrac{1}{4}\right)-\tfrac{3}{8}\pi\right)\\
&-\frac{19}{4\sqrt{15}}\left(2n^{-1}-5n^{-2}-\frac{3721}{4}n^{-3}+\frac{26\,187}{8}n^{-4}\right.\\
&\left.+\frac{5087\,026\,973}{9120}n^{-5}+\cdots\right)\sin\left(\left(n+\tfrac{1}{2}\right)\cos^{-1}\left(\tfrac{1}{4}\right)-\tfrac{3}{8}\pi\right)\Bigg].
\end{aligned}
\tag{41}
$$

The presence of singularities at $s = \pm\mathrm{i}$ near to those at $s = \exp(\pm 2\mathrm{i}\pi/3)$ and $s = (1\pm\mathrm{i}\sqrt{15})/4$ makes the subtraction (41) rather less effective than the corresponding subtraction (29) for $T < T_\mathrm{c}$ and so a further smoothing is essential. Define the operator D_c as the product of the mappings $g_n \to (g_{n-1} + g_n + g_{n+1})/3$ and $g_n \to (2g_{n-1} - g_n + 2g_{n+1})/3$. Then a useful least-squares procedure can be based on the residual

$$R_n = n^{-2} D_\mathrm{c}^2 n^{19/4} D_a^3 n^3 D_a^3 n^{13/4} g_n. \tag{42}$$

A particular fit, following the subtraction of the first term in equation (24) through $\mathrm{O}(\omega_f^{53/4})$ and that indicated in equation (41), is the antiferromagnetic scaling function in equation (36) and

$$
\begin{aligned}
\tau \ln|\tau| g_\tau^{(0)} g_\tau^{(2)} \approx \omega_f \ln|\omega_f| (1 &+ 0.994\,\underline{476\,313\,1077}\omega_f + 1.2\underline{53\,033\,259\,05}\omega_f^2\\
&+\underline{7.192\,851\,433}\omega_f^3 + \underline{89.113\,1162}\omega_f^4 + \underline{661.227\,86}\omega_f^5 + \underline{2371.200}\omega_f^6\\
&+\underline{3177.19}\omega_f^7).
\end{aligned}
\tag{43}
$$

The resulting residuals R_n of equation (42) are $|R_n| < 1.3$ on the fitting interval $76 \leqslant n \leqslant 106$. The estimated $g_\tau^{(2)}$ from (43) is

$$g_\tau^{(2)} \approx 1 + 0.494\underline{5}\tau + 0.4\underline{5}\tau^2 + \underline{6}\tau^3 + \cdots \qquad (T > T_\mathrm{c}) \tag{44}$$

which is different from equation (32) deduced for $T < T_\mathrm{c}$. The second term in each estimate is close to $\frac{1}{2}\tau$ but the differences are significant. If either fit is forced to accommodate this value, or the fits are forced to a common linear τ coefficient, the residuals increase by more than two orders of magnitude and no sensible result is obtained. It is the observation that no reasonable *common* linear τ coefficient can be found which is the basis for my conclusion that terms from irrelevant variables must be present in the susceptibility. Stated another way, it is not possible to preserve the scaling field hypothesis in its simplest form in which one both maintains the analyticity of the unknown $g_\tau^{(2)}$ and at the same time keeps without change the known $g_\tau^{(0)}$ and $g_h^{(1)}$.

It has by no means been determined that the corrections to scaling from irrelevant variables is of the form $\tau^2 \ln|\tau|$ indicated by equations (32) and (44). In fact, because the coefficients of ω_f^n in equation (43) are growing so rapidly with order it is almost certain that the corrections are not of this form. As a simple test of this idea I have assumed $g_\tau^{(2)} = 1 + \frac{1}{2}\tau + \cdots$ but allowed a modification of the amplitude of the coefficient of τ^4 in $g_h^{(1)}$. A particular fit in this case with $-4333\tau^4/98\,304$ in equation (23) replaced by $-(4333 + 127.4)\tau^4/98\,304$ is the antiferromagnetic equation (37) and

$$\tau \ln|\tau| g_\tau^{(0)} g_\tau^{(2)} \approx \omega_f \ln|\omega_f|(1 + \omega_f + 0.955\,\underline{574\,346\,23}\omega_f^2 + 0.2\underline{10\,333\,607}\omega_f^3 - \underline{6.794\,8903}\omega_f^4 - \underline{39.717\,263}\omega_f^5 - \underline{110.770\,73}\omega_f^6 - \underline{119.8342}\omega_f^7. \quad (45)$$

The two fits are comparable as determined by the magnitude of the residuals R_n, but the more reasonable growth in coefficients in equation (45) suggests that it is the more plausible representation. A very similar replacement in the low-temperature case, namely $-4333\tau^4/98\,304 \to -(4333 + 108.5)\tau^4/98\,304$ in equation (23), allows $g_\tau^{(2)} = 1 + \frac{1}{2}\tau + \cdots$ and a fit with residues comparable to those obtained for equation (31).

In conclusion, no reasonable fit of the susceptibility series is possible with the nonlinear scaling field equation (24) in the absence of irrelevant scaling variables. The irrelevant variables could contribute already at order $\tau^2 \ln|\tau|$, making the determination of $g_\tau^{(2)}$ beyond the trivial $g_\tau^{(2)} = 1$ ambiguous. A more plausible situation is that the irrelevant variables first contribute at order $\tau^{9/4}$ in which case $g_\tau^{(2)} = 1 + \frac{1}{2}\tau + \cdots$ is very likely. Whether the irrelevant variables might be incorporated into scaling field corrections of order $\tau^4 \ln|\tau|$ as suggested by Barma and Fisher [10] in their generalized scaling field proposal is not yet clear. Certainly the additive constants in the amplitude shifts $4333 \to 4333 + 127.4$ and $4333 \to 4333 + 108.5$ discussed in the previous paragraph might very well be approximations to $A_\pm \ln|B_\pm \tau|$ with $A_\pm$, $B_\pm$ constants applicable for $T \gtrless T_c$. However, what needs to be investigated carefully, but is beyond the scope of the present paper, is whether the *same* constants enter in both temperature regimes.

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Appendix A

Series for $\chi^{(N)}$ can be generated numerically from equation (2) exactly as in I except that with the use of equation (4) there is no need to keep intermediate series of a length longer than the final output series. There is also an improvement possible due to the explicit $N!$ permutation symmetry that can be incorporated without difficulty. The number of integration points exclusive of symmetry considerations is unchanged. A rough timing estimate for an order s^n (or s^{-n}) calculation, based on the observation that $\approx 8N$ series multiplications/divisions are required in the innermost do-loop of the $(N-1)$-dimensional integration program, is

$$T \approx 8N\tau((n - N^2)^2/2)((n + 2 - N(N-1))^{N-1}/N!) \quad \text{(A1, I.52)}$$

with τ the time for a scalar multiply.

Series for the low-temperature phase are as follows. For $N = 2$, the analytical formula

$$\chi^{(2)} = ((1 + s^{-4})\boldsymbol{E}(s^{-2}) - (1 - s^{-4})\boldsymbol{K}(s^{-2}))/(3\pi(1 - s^{-2})(1 - s^{-4})^{3/4}) \tag{A2}$$

where $\boldsymbol{E}(k)$ and $\boldsymbol{K}(k)$ are complete elliptic integrals of modulus k, is given in [2, 4] and can be checked from equations (1)–(4). For $N = 4$,

$\chi^{(4)} = 16/(2s)^{16}$[1, 0, 34, 4, 816, 184, 17 032, 5528, 330 410, 137 616, 6133 502, 3080 684,
110 614 188, 64 440 400, 1955 049 704, 1286 396 624, 3404 9564 812,
24 823 080 048, 586 445 963 472, 466 880 805 208, 10 013 318 250 144,
8607 752 182 240, 169 795 200 652 544, 156 191 842 299 264,
2863 066 302 852 100, 2797 674 466 067 936, 48 052 805 813 499 830,
49 576 416 445 259 516, 803 364 601 771 139 428, 870 640 463 287 642 624,
13 386 533 329 957 780 008, 15 173 111 683 909 087 968,
222 427 274 992 084 420 564, 262 694 289 615 880 732 224,
3686 688 145 743 255 403 920, 4522 158 955 326 355 312 064,
60 974 369 623 968 898 880 496, 77 458 850 042 678 600 386 272,
1006 541 602 369 530 330 457 600, 1320 942 889 551 199 319 187 936,
16 587 553 491 378 418 040 825 784, 22 438 855 031 194 992 135 981 728,
272 946 365 719 021 571 144 410 848, 379 842 400 422 024 471 931 467 096,
4485 213 787 148 282 612 479 813 936,
6409 848 064 169 377 405 155 348 000,
73 613 386 315 722 756 079 612 977 472,
107 861 635 250 940 020 997 598 333 376,
1206 833 646 078 540 650 844 079 921 776,
1810 408 317 297 117 041 825 222 831 488] + $O(1/s^{116})$

where every nth term in square brackets is understood to be divided by $(4s^2)^{n-1}$. In the same notation,

$\chi^{(6)} = 64/(2s)^{36}$[1, 0, 70, 4, 2908, 324, 93 600, 15 236, 2582 208, 545 744, 64 243 876,
16 530 604, 1484 638 788, 446 674 112, 32 470 021 016, 11 111 354 108,
680 588 629 015, 259 652 450 776, 13 793 531 185 122, 5778 875 313 592,
272 056 612 645 156, 123 700 557 739 324, 5247 322 577 116 088,
2565 114 296 089 748, 99 339 631 383 591 880, 51 810 149 151 875 664,
1851 287 082 355 848 704, 1023 589 372 712 797 784,
34 040 288 011 272 234 608, 19 846 370 622 152 535 036,
618 712 428 348 277 070 244, 378 647 369 952 195 878 300,
11 133 211 679 566 279 650 584, 7124 004 145 333 331 471 064,

198 578 882 997 030 212 999 128, 132 409 312 519 446 049 990 112,
3514 642 813 193 500 314 740 220, 2434 766 766 257 642 423 738 800,
61 779 931 915 842 789 393 162 552,
44 348 275 917 143 635 236 792 348] + $O(1/s^{116})$

$\chi^{(8)} = 256/(2s)^{64}$[1, 0, 126, 0, 8760, 4, 444 740, 584, 18 429 842, 46 440, 661 181 352,
2666 700, 21 284 489 876, 123 775 920, 629 590 702 560, 4931 432 616,
17 399 645 554 608, 174 978 276 504, 454 782 728 642 990, 5666 937 855 136,
11 346 176 762 954 496, 170 457 295 696 784, 272 130 763 033 866 776,
4823 358 665 721 664, 6310 173 702 555 839 080,
129 653 155 664 625 380] + $O(1/s^{116})$

$\chi^{(10)} = 1024/(2s)^{100}$[1, 0, 198, 0, 20 888, 4, 1560 492, 868] + $O(1/s^{116})$.

As a check of transcription errors, note the sum of all coefficients in each square brackets grouping is 3210 306 336 610 319 288 037 819 026 449 for $\chi^{(4)}$, 112 427 917 104 490 865 032 248 448 for $\chi^{(6)}$, 6728 776 294 882 212 939 for $\chi^{(8)}$ and 1582 451 for $\chi^{(10)}$.

Series coefficients in the high-temperature phase to supplement those in I are

$\chi^{(3)} = 4(s/2)^8$[1, ..., 170 978 340 515 589 313 718 120, 335 060 480 205 265 606 068 840,
684 602 608 103 977 440 609 332, 1381 706 579 997 262 133 939 892,
2828 608 366 958 598 297 227 468, 5557 101 341 494 935 415 636 732,
11 338 147 111 979 382 845 549 828, 22 866 103 814 018 400 451 961 844,
46 732 824 738 185 549 695 640 488, 92 008 743 010 682 180 233 831 048,
187 483 755 779 768 696 740 591 852, 377 868 579 896 643 784 209 024 172,
771 120 554 264 877 537 844 229 224, 1521 009 770 708 608 999 364 253 584,
3095 812 279 834 146 303 984 008 100,
6236 036 570 504 302 100 769 258 980,
12 708 953 203 092 717 183 163 924 408,
25 108 690 989 496 635 685 935 780 792,
51 053 020 094 654 748 250 340 561 852,
102 787 677 664 076 848 815 125 060 924,
209 231 893 932 355 980 743 629 040 196,
413 953 956 242 443 810 772 751 461 844,
840 904 784 134 290 838 248 791 930 540,
1692 329 718 083 472 057 667 718 617 852,
3441 166 987 124 800 333 795 987 767 384,
6816 488 155 523 045 803 794 701 512 312,

13 835 519 566 242 086 716 131 210 779 876,
27 833 775 136 584 542 649 229 756 844 196,
56 542 048 989 347 839 083 415 540 355 048,
112 123 381 028 021 896 618 604 264 970 184,
227 404 951 925 946 698 461 028 681 867 708,
457 333 093 960 292 542 055 342 125 777 916,
928 229 378 909 233 592 378 539 505 465 320] + $O(s^{117})$

$\chi^{(5)} = 16(s/2)^{24}$[1, . . . , 51 618 661 720 552 233 864, 34 914 520 911 575 089 428,
36 052 409 051 671 379 304, 79 055 035 337 786 390 400,
867 542 726 971 353 662 636, 609 243 496 914 657 871 536,
653 488 652 464 102 895 964, 1418 334 964 232 666 691 976,
14 518 616 908 164 075 938 428, 10 556 972 362 883 121 049 256,
11 719 867 457 568 418 427 552, 25 213 740 500 370 016 547 184,
242 084 144 571 191 426 200 055, 181 825 689 673 000 784 234 256,
208 261 303 624 861 442 714 088, 444 642 172 195 199 815 656 320,
4023 699 374 843 213 608 022 380, 3115 093 947 491 274 383 904 944,
3671 185 631 869 949 162 158 912, 7786 107 557 833 567 594 472 140,
66 692 726 260 805 066 384 497 404, 53 120 715 346 208 833 152 690 176,
64 259 981 130 138 513 331 246 464, 135 494 241 796 199 518 425 986 968,
1102 738 110 024 561 838 992 358 444, 902 123 340 477 352 211 230 691 464,
1117 808 998 097 154 782 047 821 804, 2344 834 401 280 243 812 497 116 256,
18 194 138 527 651 859 775 499 676 748,
15 264 191 780 788 899 772 577 124 824,
19 336 984 898 789 052 467 071 047 320,
40 378 538 454 184 718 190 567 490 576,
299 613 485 835 811 943 745 832 462 524] + $O(s^{117})$

$\chi^{(7)} = 64(s/2)^{48}$[1, . . . , 81 610 343 951 508, 1466 844 589 636, 663 120 891 668,
29 321 249 606 976, 1797 514 276 437 711, 38 392 007 711 240,
19 244 414 195 012, 711 042 350 579 476, 38 364 678 187 366 272,
954 028 075 305 788, 522 351 335 517 580, 16 536 532 709 907 144,
797 298 354 953 513 700, 22 707 413 725 214 080, 13 422 042 208 702 728,
371 298 047 779 473 440, 16 196 037 171 229 162 872,
521 202 484 651 584 116, 329 485 078 154 416 456, 8091 036 110 069 443 644,

322 583 813 599 182 464 183, 11 598 350 872 008 924 200,
7781 835 176 201 516 652, 171 833 490 340 034 513 148,
6315 746 844 925 803 912 700, 251 300 917 168 878 143 316,
177 820 458 618 682 617 680, 3568 703 069 836 131 909 620,
121 805 723 084 988 763 487 569, 5319 976 394 951 585 446 820,
3948 970 248 129 057 475 872, 72 681 911 457 169 995 752 756,
2318 112 680 138 856 429 317 460] + $O(s^{117})$

$\chi^{(9)} = 256(s/2)^{80}$[1, 0, 0, 0, 160, 0, 0, 0, 13 839, 4, 0, 4, 858 704, 708, 0, 728, 42 821 009,
67 252, 44, 70 976, 1823 591 632, 4553 260, 8552, 4924 124, 68 849 090 530,
246 325 612, 886 592, 272 571 000, 2362 955 813 664, 11 321 647 420,
65 153 952, 12 801 733 500, 75 046 023 962 279, 459 187 169 680,
3806 965 664, 529 929 285 744, 2234 769 195 868 396] + $O(s^{117})$.

The sum of the coefficients listed in each square brackets grouping above is

1 + · · · + 1836 920 256 588 048 308 225 813 577 267 856 for $\chi^{(3)}$
1 + · · · + 398 594 150 670 908 937 375 825 751 595 for $\chi^{(5)}$
1 + · · · + 2532 723 031 648 180 443 053 023 for $\chi^{(7)}$

and

2313 266 533 385 030 for $\chi^{(9)}$.

Appendix B

The transfer matrix spectral formalism was developed by Abraham in a series of papers culminating in a description of pair correlations [22] and the general n-point function [23]. However, the clearest statement of the final pair formulae, in the high-temperature phase $s < 1$, appears in [15] and I will use this reference exclusively, with equations from it given as (**DBA). The equivalence of the susceptibility derived by Abraham and that discussed in I and this paper will be established if one can show

$$|F^x((e^{i\phi})_{2n+1})|^2 \stackrel{?}{=} (1-s^4)^{1/4} s^{-1} (G^{(2n+1)})^2 \Big/ \prod_{i=1}^{2n+1} \sinh\gamma_i \tag{B1}$$

where G is the antisymmetric sum (5) and F^x is defined by equations (22)–(27DBA). To begin, note that

$$\begin{aligned} \sinh\gamma &= ((s+s^{-1}-\cos\phi)^2-1)^{1/2} \\ &= ((A-z)(B^{-1}-z)(A-z^{-1})(B^{-1}-z^{-1})B/(4A))^{1/2} \end{aligned} \tag{B2}$$

where $z = \exp(i\phi)$ and, for $s < 1$, the A and B^{-1} are the branch points outside the unit circle $|z| = 1$; explicitly

$$\begin{aligned} A &= \coth(K)\exp(2K) = s^{-1}(s+(1+s^2)^{1/2})(1+(1+s^2)^{1/2}) \\ B &= \tanh(K)\exp(2K) = s(s+(1+s^2)^{1/2})/(1+(1+s^2)^{1/2}) \end{aligned} \tag{B3}$$

as given in (27DBA). The Onsager hyperbolic function can be factored as

$$\sinh\gamma = f(z)f(z^{-1})/(4A/B)^{1/2} \qquad 1/\sinh\gamma = (4A/B)^{1/2}g(z)g(z^{-1}) \tag{B4}$$

with $f(z) = ((A-z)(B^{-1}-z))^{1/2} = 1/g(z^{-1})$ given in (25)–(26DBA). It is to be understood that all square roots are positive real for $z = -1$.

With these preliminary results in hand, one can easily show that the contraction function (25DBA) can be rewritten as

$$f_-(z_i, z_j) = (4A/B)^{1/2}g(z_i)g(z_j)(\sinh\gamma_i - \sinh\gamma_j)z_iz_j/(z_iz_j - 1) \tag{B5}$$

while the generator (22DBA) factors into

$$F^x((z)_{2n+1}) = -(1-s^4)^{1/8}(4A/B)^{n/2}\prod_{i=1}^{2n+1} g(z_i)\sum_1^{2n+1}\mathrm{Pf}\{f_{ij}(\mathrm{DBA})\} \tag{B6}$$

with the Pfaffian elements $f_{ij}(\mathrm{DBA}) = (\sinh\gamma_i - \sinh\gamma_j)z_iz_j/(z_iz_j-1)$ from (B5). The Pfaffian sum in (B6) is exactly that in equation (5) except for the difference in elements. In view of the fact that only $|F^x|^2$ appears in (B1) one can modify the Abraham element $f_{ij}(\mathrm{DBA})$ to

$$\begin{aligned} f_{ij}(\mathrm{DBA}) &= \mathrm{i}(\sinh\gamma_i - \sinh\gamma_j)z_iz_j/(z_iz_j-1) \\ &\equiv \mathrm{i}\tfrac{1}{2}(\sinh\gamma_i - \sinh\gamma_j)(z_iz_j+1)/(z_iz_j-1) \\ &= \tfrac{1}{2}(\sinh\gamma_i - \sinh\gamma_j)\cos\tfrac{1}{2}(\phi_i+\phi_j)/\sin\tfrac{1}{2}(\phi_i+\phi_j) = f_{ij}. \end{aligned} \tag{B7}$$

The equivalence ($\equiv$) in (B7) follows because the difference of the first two lines is the function $\mathrm{i}\frac{1}{2}(\sinh\gamma_i - \sinh\gamma_j)$ which vanishes in any antisymmetric sum such as (5) that cycles over all indices; this is the cumulant property discussed in I and can be proved inductively. The final equality to f_{ij} of equation (6) follows trivially from the identity $\sinh\gamma_i - \sinh\gamma_j = 2\cosh\frac{1}{2}(\gamma_i+\gamma_j)\sinh\frac{1}{2}(\gamma_i-\gamma_j)$. On taking the square of (B6) and using the factorization of $1/\sinh\gamma$ in (B4) one finds that

$$|F^x((\mathrm{e}^{\mathrm{i}\phi})_{2n+1})|^2 = (1-s^4)^{1/4}(B/(4A))^{1/2}(G^{(2n+1)})^2\Big/\prod_{i=1}^{2n+1}\sinh\gamma_i \tag{B8}$$

so that (B1) is verified *except* for a factor $s(B/(4A))^{1/2} = \sinh^2(K)$. I have not traced the source of the $\sinh(K)$ error in F^x.

For a comparison at low temperatures, $s > 1$, I refer to [22, 23]. The equivalence that can be verified is

$$|F^x((\mathrm{e}^{\mathrm{i}\phi})_{2n})|^2 = (1-s^{-4})^{1/4}(\mathrm{Pf}\{h_{ij}\})^2\Big/\prod_{i=1}^{2n}\sinh\gamma_i \tag{B9}$$

where the h_{ij} are the elements in equation (4), while the left-hand side of (B9) is defined by

$$\begin{aligned} &F^x((\mathrm{e}^{\mathrm{i}\phi})_{2n}) = (1-s^{-4})^{1/8}\mathrm{Pf}\{f_-(z_i,z_j)\} \\ &f_-(z_i,z_j) = (f(z_i)/f(z_j^{-1}) - f(z_j)/f(z_i^{-1}))z_iz_j/(z_iz_j-1) \\ &f(z) = ((A-z)/(B-z))^{1/2}. \end{aligned} \tag{B10}$$

A and B are still given by (B3); the difference with the high-temperature case is that now B is the branch point of $\sinh\gamma$ that is outside the circle $|z| = 1$. This in turn modifies the function

$f(z)$ that plays a role in the Wiener–Hopf factorization in the spectral approach. Algebraic reduction of the Pfaffian elements $f_-(z_i, z_j)$ with the help of the identities

$$((A-z)(A-z^{-1}))^{1/2} = 2\sqrt{A}\cosh\tfrac{1}{2}\gamma$$
$$((B-z)(B-z^{-1}))^{1/2} = 2\sqrt{B}\sinh\tfrac{1}{2}\gamma \tag{B11}$$

leads to

$$f_-(z_i, z_j) = \mathrm{i}\exp\tfrac{1}{2}\mathrm{i}(\phi_i+\phi_j)h_{ij}/((A-z_i^{-1})(A-z_j^{-1})(B-z_i)(B-z_j)/(4AB))^{1/2}. \tag{B12}$$

Further simplification is possible because only $|F^{x|2}$ enters into the equivalence equation (B9). For example, the factor $\exp\frac{1}{2}\mathrm{i}(\phi_i+\phi_j)$ in (B12), which contributes to a phase $\exp\frac{1}{2}\mathrm{i}\sum_1^{2n}\phi_i$ common to every term in the Pfaffian in (B10), can be dropped. Also the denominator in (B12) can be replaced by its absolute value on the circle $|z| = 1$; one finally obtains

$$f_-(z_i, z_j) \equiv h_{ij}/(\sinh\gamma_i \sinh\gamma_j)^{1/2} \tag{B13}$$

and (B9) is proved.

Appendix C

The formulae given below enable one to transform the elliptic representation of Yamada [4–6] to trigonometric/hyperbolic form. They are essentially the Onsager [7] and Yang [14] transformations but it is convenient to have them collected together. Also, this listing can be used to eliminate a potential source of confusion as a result of the differences in the convention adopted for the argument of the Jacobi functions $\mathrm{sn}\,u$, $\mathrm{cn}\,u$, $\mathrm{dn}\,u$, etc. In the following it is understood that a full period of $\mathrm{sn}\,u$ and $\mathrm{cn}\,u$ on the real axis is $0 \leqslant u \leqslant 4K$ (Yamada has used both this definition of u and u such that sn has period $0 \leqslant u \leqslant 2K$).

The starting point of the Onsager/Yang transformations can be taken to be the functional relations

$$\exp(\pm\tfrac{1}{2}\mathrm{i}\phi - \tfrac{1}{2}\gamma) = \sqrt{k}\,\mathrm{sn}\,\tfrac{1}{2}(u \pm \mathrm{i}a) \qquad \mathrm{sn}\,\mathrm{i}a = \mathrm{i}/\sqrt{k} \tag{C1}$$

with k the modulus of the complete elliptic integral K and given by $k = s^2$ ($k = 1/s^2$) for $T > T_c$ ($T < T_c$). The parameter a is real; the point $u = 0$ is $\phi = \pi$; the point $\phi = 0$ is $u = 2K$. From (C1) one can derive

$$z = \exp(\mathrm{i}\phi) = \mathrm{sn}\,\tfrac{1}{2}(u+\mathrm{i}a)/\,\mathrm{sn}\,\tfrac{1}{2}(u-\mathrm{i}a) = k\,\mathrm{sn}\,\tfrac{1}{2}(u+\mathrm{i}a)\,\mathrm{sn}\,\tfrac{1}{2}(u-\mathrm{i}a+2\mathrm{i}K')$$
$$= -(\mathrm{cn}\,u - \mathrm{i}(1+k)^{1/2}\,\mathrm{sn}\,u)/(\mathrm{dn}\,u + \mathrm{i}(k+k^2)^{1/2}\,\mathrm{sn}\,u) \tag{C2}$$
$$x = \exp(-\gamma) = k\,\mathrm{sn}\,\tfrac{1}{2}(u+\mathrm{i}a)\,\mathrm{sn}\,\tfrac{1}{2}(u-\mathrm{i}a) = ((k+k^2)^{1/2} - k\,\mathrm{cn}\,u)/((1+k)^{1/2}+\mathrm{dn}\,u)$$

and from these in turn

$$\cos\phi = (\sqrt{k}\,\mathrm{dn}\,u - \mathrm{cn}\,u)/D \qquad \sin\phi = \mathrm{sn}\,u(1-k)(1+k)^{1/2}/D$$
$$\cosh\gamma = (\mathrm{dn}\,u/\sqrt{k} - k\,\mathrm{cn}\,u)/D \qquad y^{-1} = \sinh\gamma = (1-k)(1+1/k)^{1/2}/D \tag{C3}$$
$$\mathrm{d}\phi/\mathrm{d}u = -(1-k)(1+k)^{1/2}/D \qquad D = \mathrm{dn}\,u - \sqrt{k}\,\mathrm{cn}\,u$$

which are some of the formulae given by Onsager [7]. The necessary identity $\cosh\gamma + \cos\phi = \sqrt{k} + 1/\sqrt{k}$ is clearly satisfied. The integral transformation is

$$\int_{-\pi}^{\pi} \mathrm{d}\phi\, y \ldots = \sqrt{k}\int_0^{4K} \mathrm{d}u \ldots \tag{C4}$$

and

$$h_{ij} = \sin\tfrac{1}{2}(\phi_i - \phi_j)/\sinh\tfrac{1}{2}(\gamma_i + \gamma_j) = -\sqrt{k}\,\mathrm{sn}\,\tfrac{1}{2}(u_i - u_j). \tag{C5}$$

All equations (C2)–(C5) follow from (C1) and standard identities for the Jacobi elliptic functions [24]. Many other forms of the identities can be given; it is not easy to spot equivalences and indeed even to verify, let alone derive, the formulae above I have found algebraic packages such as *Maple* useful.

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